Relative Gain Array Analysis for Uncertain Process Models

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Relative gain array (RGA) analysis has been widely used in process control to identify promising control structures and to characterize the degree of process interactions between controlled and manipulated variables. However, the influence of process model uncertainty on RGA analysis has received little attention. Analytical expressions for RGA uncertainty bounds are derived for 2×2 control problems and for general, $n \times n$ control problems. Both worst-case bounds and statistical uncertainty bounds are derived. Several simulation examples illustrate the new results. The RGA uncertainty bounds provide useful information concerning model accuracy requirements and the robustness of decentralized control systems.

Introduction

RGA was introduced by Bristol (1966) as a measure of process interactions in multi-input, multi-output control problems. Due to its simplicity and utility, the RGA analysis has been widely used to identify promising decentralized (multi-loop) control systems based on limited information, steady-state gains. The books by McAvoy (1983a) and Shinskey (1988) describe many practical applications. Further research has shown that the RGA is related to many fundamental closed-loop system properties such as stability, robustness, decentralized integral controllability, and failure tolerance (Grosdidier et al., 1985; Yu and Luyben, 1987; Skogestad and Morari, 1987; Chiu and Arkun, 1991; Hovd and Skogestad, 1992; Chen et al., 1994; Zhu and Jutan, 1996; Lee and Edgar, 2000). The related concept of relative disturbance gain (RDG) has been proposed (Stanley et al., 1985) and extended (Chang and Yu, 1992, 1994) in order to evaluate the effect of disturbances. Because the original RGA analysis was based on only steady-state information, it does not consider the dynamic behavior of the process in response to disturbances and set point changes. Consequently, dynamic versions of the RGA have been proposed (Witcher and McAvoy, 1997; Bristol, 1978; Tung and Edgar, 1981; Gagnepain and Seborg, 1982; Jensen et al., 1986; McAvoy et al., 2001).

Although the process control literature is replete with analyses of RGA properties based on process models, the effect of model uncertainty on RGA analysis has received little attention. There are many factors in practice that can contribute to model uncertainty such as plant/model mismatch, changes in operating conditions, drift of physical parameters, and so on. Thus, process models are never perfect. For uncertain plant models, an RGA analysis based on a nominal process model can lead to incorrect conclusions. Consequently, it is important to know how sensitive the RGA analysis is to model uncertainty.

In this article, analytical expressions for RGA uncertainty bounds are derived for two important classes of model uncertainty descriptions: worst-case bounds (hard limits) and statistically-based bounds (such as three sigma limits). New results are developed for both 2×2 control problems and general $n \times n$ control problems.

Definition and Properties of RGA

For an $n \times n$ system, Bristol (1966) defined each element in the RGA as the ratio of the open-loop gain with all other loops open to the closed-loop gain with all other loops closed:

$$\lambda_{ij} = \frac{\left(\partial y_i/\partial u_j\right)_{u_{k\neq j}}}{\left(\partial y_i/\partial u_j\right)_{y_{k\neq j}}} = \frac{\text{open-loop gain}}{\text{closed-loop gain}} \tag{1}$$

λ_{ij} is the relative gain between y_i, the ith controlled variable, and u_j, the jth manipulated variable. The RGA matrix $\boldsymbol{\Lambda}$ can be written in terms of the steady-state gain matrix $\boldsymbol{K} = \{K_{ij}\}$, and the transpose of its inverse $[\boldsymbol{K}^{-1}]^T = \{K'_{ij}\}$

$$\boldsymbol{\Lambda} = \{\lambda_{ij}\} = \{K_{ij}K'_{ij}\} = \boldsymbol{K} \otimes [\boldsymbol{K}^{-1}]^T \tag{2}$$

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where $\otimes$ denotes element-by-element multiplication. A dynamic version of the RGA can be obtained by replacing the steady-state gain matrix $\boldsymbol{K}$ in Eq. 2 with the corresponding transfer function matrix $\boldsymbol{G}(s)$, and then substituting $s = j\omega$ (McAvoy, 1983b; Grosdidier et al., 1985). Thus the steady-state RGA is the limiting case of the dynamic RGA as $s \to 0$. The RGA matrix $\boldsymbol{\Lambda}$ has the following properties (Seborg et al., 1989; Skogestad and Postlethwaite, 1996):

(1) The sum of the elements in each row or column is one.

(2) The RGA elements are dimensionless and thus not affected by choice of units or scaling of variables.

(3) Any permutation of rows and columns in the $\boldsymbol{K}$ matrix results in the same permutation in the RGA.

Both the steady-state and the dynamic RGA have these properties. In this article, for simplicity of notation, only the steady-state RGA is considered. However, the results can easily be generalized to the dynamic RGA.

The effect of model uncertainty on RGA analysis has received surprisingly little attention. Grosdidier et al. (1985) derived an expression for the local sensitivity of λ_{ij} to changes in the corresponding process gain K_{ij}

$$\frac{d\lambda_{ij}}{\lambda_{ij}} = \left(1 - \lambda_{ij}\right)\frac{dK_{ij}}{K_{ij}} \tag{3}$$

In a derivation based on this expression, Yu and Luyben (1987) showed that the $\boldsymbol{K}$ will become singular if a single, arbitrary element K_{ij} is perturbed by the following amount

$$\Delta K_{ij} = -\frac{K_{ij}}{\lambda_{ij}} \tag{4}$$

This perturbation provides an upper limit on the maximum change in a single process gain that will maintain integral controllability (Yu and Luyben, 1987). However, Eqs. 3 and 4 are based on the restrictive assumption that only one process gain changes. Thus, these results are not applicable to the more realistic situation where uncertainties are associated with more than one process gain. Furthermore, the element-by-element uncertainties are usually correlated (Skogestad, 1992).

Grosdidier et al. (1985) have derived a more general result for integral controllability. If $\hat{\boldsymbol{K}}$ is a nominal gain matrix and $\| \boldsymbol{\Lambda} \|_1$ and $\| \boldsymbol{\Lambda} \|_\infty$ are matrix norms for the $\boldsymbol{\Lambda}(\hat{\boldsymbol{K}})$, then integral controllability will be preserved if the perturbed gain matrix $\boldsymbol{K}$ satisfies the following bound

$$\frac{\| \boldsymbol{K} - \hat{\boldsymbol{K}} \|}{\| \hat{\boldsymbol{K}} \|} \le \frac{1}{2\max\left(\| \boldsymbol{\Lambda} \|_1, \| \boldsymbol{\Lambda} \|_\infty \right)} \tag{5}$$

While Eq. 5 is an important result, it does address the principal concern of this article: to what extent does a prescribed degree of uncertainty in $\hat{\boldsymbol{K}}$ result in uncertainty in the calculated RGA and analysis based on it?

In this article, uncertainty bounds for the RGA matrix are derived based on general classes of model uncertainty that include correlated uncertainties for the elements of the $n \times n$ gain matrix.

Analysis for 2×2 Control Problems

Consider a control problem with two controlled variables and two manipulated variables. The steady-state gain matrix for this 2×2 control problem is

$$\boldsymbol{K} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \tag{6}$$

and the corresponding RGA matrix is

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_{11} & 1 - \lambda_{11} \\ 1 - \lambda_{11} & \lambda_{11} \end{bmatrix} \tag{7}$$

where

$$\lambda_{11} = \frac{1}{1 - \kappa} \tag{8}$$

$$\kappa \triangleq \frac{K_{12}K_{21}}{K_{11}K_{22}} \tag{9}$$

and κ is referred to as the interaction quotient or interaction measure (Rijnsdorp, 1965). Clearly, $\kappa = 1$ is a singular point for the RGA matrix.

Because the RGA matrix depends on the steady-state gains, model uncertainty in $\boldsymbol{K}$ produces uncertainties in the calculated RGA. In this article, two types of model uncertainty descriptions are considered.

Worst-case bounds for steady-state gains

Suppose that symmetric upper and lower bounds are available for each steady-state gain, that is, the unknown steady-state gain K_{ij} varies within a range of $\pm \Delta K_{ij}$ around a nominal value, $\hat{K}_{ij}$. The corresponding range for the interaction quotient κ will be denoted as

$$\kappa^l \le \kappa \le \kappa^h \tag{10}$$

where

$$\kappa^1 = \min \frac{K_{12}K_{21}}{K_{11}K_{22}} \tag{11}$$

$$\kappa^h = \max \frac{K_{12}K_{21}}{K_{11}K_{22}} \tag{12}$$

If the range of κ in Eq. 10 does not include $\kappa = 1$, then the range for λ_{11} is

$$\frac{1}{1 - \kappa^l} \le \lambda_{11} \le \frac{1}{1 - \kappa^h} \tag{13}$$

If the range of κ does include $\kappa = 1$, then the sign of λ_{11} changes at the singular point where $\kappa = 1$ and the uncertainty range for λ_{11} is

$$-\infty \le \lambda_{11} \le \frac{1}{1 - \kappa^h} \quad \text{and} \quad \frac{1}{1 - \kappa^l} \le \lambda_{11} \le \infty \tag{14}$$

Next, a transfer function model which has been used in many previous studies will be used to illustrate these results.

Example 1. Wood and Berry (1973) developed the following empirical model of a pilot-scale distillation column that is

used to separate a methanol-water mixture

$$\begin{bmatrix} X_D(s) \\ X_B(s) \end{bmatrix} = \begin{bmatrix} \dfrac{12.8e^{-s}}{16.7s+1} & \dfrac{-18.9e^{-3s}}{21s+1} \\ \dfrac{6.6e^{-7s}}{10.9s+1} & \dfrac{-19.4e^{-3s}}{14.4s+1} \end{bmatrix} \begin{bmatrix} R(s) \\ S(s) \end{bmatrix} \quad (15)$$

where X_D and X_B are the overhead and bottoms compositions of methanol, respectively; R is the reflux flow rate and S is the steam flow rate to the reboiler.

The nominal steady-state gain matrix is

$$\hat{\boldsymbol{K}} = \begin{bmatrix} 12.8 & -18.9 \\ 6.6 & -19.4 \end{bmatrix} \quad (16)$$

The corresponding RGA matrix and interaction quotient κ are

$$\hat{\boldsymbol{\Lambda}} = \begin{bmatrix} 2.01 & -1.01 \\ -1.01 & 2.01 \end{bmatrix} \quad (17)$$

$$\hat{\kappa} = 0.502 \quad (18)$$

which suggest the following controller pairing: $X_D - R/X_B - S$.

Assume that the uncertainty bounds for the steady-state gains are given by

$$|\Delta K_{ij}| \le \alpha |\hat{K}_{ij}| \quad (19)$$

Three different α values will be considered. Because κ^l and κ^h can be calculated using Eqs. 11 and 12, the range of λ_{11} for each value of α can be obtained.

Case 1: $\alpha = 0.01$. According to Eqs. 11 and 12, the upper and lower bounds for κ are $\kappa^l = 0.48$ and $\kappa^h = 0.52$, so that the bounds for λ_{11} are given by Eq. 13

$$1.92 \le \lambda_{11} \le 2.08$$

Thus, the recommended controller pairing from the RGA analysis is $X_D - R/X_B - S$.

Case 2: $\alpha = 0.1$. Now $\kappa^l = 0.34$ and $\kappa^h = 0.75$ and

$$1.52 \le \lambda_{11} \le 4$$

The suggested pairing is still $X_D - R/X_B - S$.

Case 3: $\alpha = 0.25$. Now $\kappa^l = 0.18$, $\kappa^h = 1.39$, and thus the range of κ includes 1. From Eq. 14 it follows that

$$-\infty \le \lambda_{11} \le -2.53 \quad \text{and} \quad 1.22 \le \lambda_{11} \le \infty$$

Thus, within the uncertainty range for $\alpha = 0.25$, the RGA matrix becomes singular and λ_{ij} changes sign. Consequently, no single controller pairing can be recommended for the entire uncertainty range because both positive and negative values of λ_{11} are possible.

In this example, if $\lambda_{11} > 0.5$ for the entire uncertainty range in Eq. 13, then the same controller pairing is recommended by the RGA analysis. According to Eq. 8, it is easy to conclude that λ_{11} is greater than 0.5 if $-1 < \kappa < 1$. For this example, κ^l will not be equal to -1 for $\alpha \le 1$, while $\alpha = 0.17$ makes $\kappa^h = 1$. Therefore, the upper bound for α is given by

$$0 \le \alpha \le 0.17 \quad (20)$$

If the model uncertainty satisfies this constraint, the corresponding λ_{11} range is always greater than 0.5 and the RGA analysis suggests the same controller pairing.

Table 1. Comparison of Uncertainty Analysis Methods

α	Uncertainty in All Elements	Uncertainty in K_{11} Only
0.01	$1.92 \le \lambda_{11} \le 2.08$	$1.99 \le \lambda_{11} \le 2.03$
0.1	$1.52 \le \lambda_{11} \le 4$	$1.81 \le \lambda_{11} \le 2.21$
0.25	$-\infty \le \lambda_{11} \le -2.53$ and $1.22 \le \lambda_{11} \le \infty$	$1.50 \le \lambda_{11} \le 2.52$

Table 1 compares the results of two uncertainty analyses for Example 1. The results for Cases 1–3 are shown as "uncertainty in all elements." The results for "uncertainty in K_{11} only" were calculated using Eq. 3. Table 1 indicates that considering only uncertainties in the corresponding gain element can produce overly optimistic estimates of the uncertainty in λ_{ij}.

This example demonstrates that model uncertainty can have a serious effect on control structure selection if the standard RGA analysis is employed. Model uncertainty should be incorporated into the RGA analysis in order to ensure the robustness of the decentralized control system.

Statistical description of uncertainty bounds

Because worst-case situations usually have a low probability of occurring, the RGA analysis of the previous section may be very conservative for more typical operating conditions that are much more probable. Thus, statistical descriptions of model uncertainty are more reasonable. They can be developed from a wide variety of sources, including statistical information for phenomenological model parameters, empirical model parameters, or the frequency response (Cloud and Kouvaritakis, 1987; Correa, 1989; Goodwin and Salgado, 1989; Stengel and Ryan, 1989; Schaper et al., 1992). Additional process knowledge is often available in the form of engineering heuristics and information about the range of operating conditions. These sources of information should also be considered when developing the uncertainty description.

Consider 2×2 systems where the nominal values of the steady-state gains and the covariance matrix are denoted by

$$\hat{\boldsymbol{K}} = \begin{bmatrix} \hat{K}_{11} & \hat{K}_{12} \\ \hat{K}_{21} & \hat{K}_{22} \end{bmatrix} \quad (21)$$

$$\Sigma(\boldsymbol{K}) \triangleq \left[\operatorname{cov}(K_{ij}, K_{mn})\right] \quad (22)$$

The nominal value $\hat{\lambda}_{ij}$ can be calculated from Eqs. 8 and 9 based on the nominal values of steady-state gain matrix $\hat{\boldsymbol{K}}$. Because λ_{ij} is a function of steady-state gains K_{kl}, it can be approximated by introducing a Taylor's series expansion and truncating after the first-order terms

$$\lambda_{ij} \approx \lambda_{ij}|_{\hat{\boldsymbol{K}}} + \sum_{k=1}^{2}\sum_{l=1}^{2} \left(\frac{\partial \lambda_{ij}}{\partial K_{kl}}\right)_{\hat{\boldsymbol{K}}} \left(K_{kl} - \hat{K}_{kl}\right) \quad (23)$$

Let $\hat{\lambda}_{ij} = \lambda_{ij}|_{\hat{K}}$. Then, it follows Eq. 23 that

$$\left(\lambda_{ij} - \hat{\lambda}_{ij}\right)^2 \approx \left[\sum_{k=1}^{2}\sum_{l=1}^{2}\left(\frac{\partial\lambda_{ij}}{\partial K_{kl}}\right)_{\hat{K}}\left(K_{kl} - \hat{K}_{kl}\right)\right]^2 \tag{24}$$

$$= \sum_{k=1}^{2}\sum_{l=1}^{2}\sum_{m=1}^{2}\sum_{n=1}^{2}\left(\frac{\partial\lambda_{ij}}{\partial K_{kl}}\right)_{\hat{K}} \times \left(\frac{\partial\lambda_{ij}}{\partial K_{mn}}\right)_{\hat{K}}\left(K_{kl} - \hat{K}_{kl}\right)\left(K_{mn} - \hat{K}_{mn}\right) \tag{25}$$

Thus, the variance of λ_{ij} can be approximated as

$$\sigma_{\lambda_{ij}}^2 = E\left[\left(\lambda_{ij} - \hat{\lambda}_{ij}\right)^2\right] \tag{26}$$

$$\approx \sum_{k=1}^{2}\sum_{l=1}^{2}\sum_{m=1}^{2}\sum_{n=1}^{2}\left(\frac{\partial\lambda_{ij}}{\partial K_{kl}}\right)_{\hat{K}}\left(\frac{\partial\lambda_{ij}}{\partial K_{mn}}\right)_{\hat{K}}\mathrm{cov}(K_{kl}, K_{mn}) \tag{27}$$

where E denotes the expectation operator. Thus, a statistical description of the uncertainty bounds for λ_{ij} has been derived based on the nominal value and covariance of $\boldsymbol{K}$.

The Wood-Berry distillation column model is used again to demonstrate the utility of statistically-based uncertainty bounds.

Example 2. Assume that the steady-state gains are reported as $\hat{K}_{ij} \pm 3\sigma_{K_{ij}}$ where the standard deviation for each steady-state gain is given by

$$\sigma_{K_{ij}} = \sqrt{\mathrm{cov}(K_{ij}, K_{ij})} = \frac{1}{3}\alpha|\hat{K}_{ij}| \tag{28}$$

and each gain is uncorrelated, that is, the covariance between different gains is zero. A comparison of Eqs. 19 and 28 indicates that $3\sigma_{K_{ij}}$ corresponds to the range of uncertainty for the worst-case analysis of the previous section. This choice was made so that the results obtained using worst-case bounds and statistical bounds can be compared. Next, three values of α will be considered.

Case 1: $\alpha = 0.01$. By using Eq. 27, the variance of λ_{11} is obtained

$$\sigma_{\lambda_{11}}^2 \approx 0.000183 \tag{29}$$

Therefore, the 3σ limits for λ_{11} can be represented as

$$\lambda_{11} = \hat{\lambda}_{11} \pm 3\sigma_{\lambda_{11}} \tag{30}$$

or

$$1.97 \le \lambda_{11} \le 2.05 \tag{31}$$

Case 2: $\alpha = 0.1$. The variance and uncertainty of λ_{11} are given by

$$\sigma_{\lambda_{11}}^2 \approx 0.0183 \tag{32}$$

or

$$1.60 \le \lambda_{11} \le 2.42 \tag{33}$$

Case 3: $\alpha = 0.25$. The variance and uncertainty of λ_{11} are given by

$$\sigma_{\lambda_{11}}^2 \approx 0.114 \tag{34}$$

or

$$0.995 \le \lambda_{11} \le 3.02 \tag{35}$$

For all three cases, the recommended controller pairing $(y_1 - u_1/y_2 - u_2)$ is valid for the entire model uncertainty range. Comparison of Examples 1 and 2 indicates that the results for the hard uncertainty bounds are much more conservative, because the worst-case situation is considered, while in practice, it may have a very low probability of actually occurring.

Model uncertainty descriptions from process identification

Statistically-based model uncertainty descriptions are often generated when process identification techniques are employed to develop empirical process models from input-output data. For example, commercial software packages produce confidence intervals for estimated model parameters and/or the frequency response of the model. However, a detailed description of these identification methods and the response uncertainty description is beyond the scope of this article. The objective of this article is to derive the uncertainty description of the RGA based on model uncertainty descriptions which have already been obtained.

The following analysis is valid for a wide class of linear, dynamic stochastic models that includes state-space models and auto-regressive, integrated moving average (ARIMA) models. However, for purposes of illustration, we will consider a particular class of models, finite impulse response (FIR) models with stochastic disturbances

$$y(t) = G(q,\boldsymbol{\Theta})u(t) + H(q,\boldsymbol{\Theta})e(t) \tag{36}$$

where

$$y(t),\ e(t) \in R^{n_y},\ u(t) \in R^{N_u} \quad \text{and}$$

$$\boldsymbol{\Theta} = [\theta_1, \ldots, \theta_n]^T \tag{37}$$

$$G(q,\boldsymbol{\Theta}) = \sum_{k=1}^{\infty} g(k,\boldsymbol{\Theta})q^{-k} \tag{38}$$

$$H(q,\boldsymbol{\Theta}) = 1 + \sum_{k=1}^{\infty} h(k,\boldsymbol{\Theta})q^{-k} \tag{39}$$

Here, $e(t)$ is a sequence of independent random variables with zero means and the same variance σ^2. Assume that $\boldsymbol{\Theta} = [\theta_1, \ldots, \theta_n]^T$ is the vector of model parameters to be identified and that $\hat{\boldsymbol{\Theta}}$ denotes an asymptotically normal and unbiased estimate of $\boldsymbol{\Theta}$. The estimated covariance matrix $\Sigma(\boldsymbol{\Theta})$ of the parameter estimation error is assumed to be obtained during process identification. Therefore

$$(\hat{\boldsymbol{\Theta}} - \boldsymbol{\Theta})^T \Sigma^{-1}(\boldsymbol{\Theta})(\hat{\boldsymbol{\Theta}} - \boldsymbol{\Theta}) \to \chi_q^2 \tag{40}$$

where q is the number of model parameters and χ_q^2 is the chi-square distribution with q degrees of freedom. This result

enables the construction of confidence bounds

$$P\left[(\hat{\boldsymbol{\Theta}}-\boldsymbol{\Theta})^T\Sigma^{-1}(\boldsymbol{\Theta})(\hat{\boldsymbol{\Theta}}-\boldsymbol{\Theta})\le\chi^2_{a,q}\right]=1-\alpha \quad (41)$$

where $1-\alpha$ is the probability that the true parameters lie within the specified ellipsoidal confidence region and $\chi^2_{a,q}$ is the upper α percentage point of χ^2_q. Let $K_{ij}(\omega,\boldsymbol{\Theta})$ be the magnitude of the transfer function from input i to output j of a 2×2 system and $\boldsymbol{K}(\omega,\boldsymbol{\Theta})=[K_{ij}(\omega,\boldsymbol{\Theta})]_{2\times2}$. The sensitivity of the function $K_{ij}(\omega,\boldsymbol{\Theta})$ to the parameters $\boldsymbol{\Theta}$ is given by the gradient

$$\boldsymbol{K}_{ij,\boldsymbol{\Theta}}(\omega,\boldsymbol{\Theta})=\frac{\partial}{\partial\boldsymbol{\Theta}^T}K_{ij}(\omega,\boldsymbol{\Theta}) \quad (42)$$

which is considered to be a $1\times n$ row vector. Define a $4\times n$ matrix $\boldsymbol{K}_{\boldsymbol{\Theta}}$ as

$$\boldsymbol{K}_{\boldsymbol{\Theta}}=\begin{bmatrix}\boldsymbol{K}_{11,\boldsymbol{\Theta}}(\omega,\boldsymbol{\Theta})\\ \boldsymbol{K}_{21,\boldsymbol{\Theta}}(\omega,\boldsymbol{\Theta})\\ \boldsymbol{K}_{12,\boldsymbol{\Theta}}(\omega,\boldsymbol{\Theta})\\ \boldsymbol{K}_{22,\boldsymbol{\Theta}}(\omega,\boldsymbol{\Theta})\end{bmatrix} \quad (43)$$

The error in estimation of $K_{ij}(\omega,\hat{\boldsymbol{\Theta}})$ can be approximated by a truncated Taylor series expansion

$$K_{ij}(\omega,\hat{\boldsymbol{\Theta}})-K_{ij}(\omega,\boldsymbol{\Theta})\approx\boldsymbol{K}_{ij,\boldsymbol{\Theta}}(\omega,\boldsymbol{\Theta})(\hat{\boldsymbol{\Theta}}-\boldsymbol{\Theta}) \quad (44)$$

The covariance matrix of $\boldsymbol{K}(\omega,\boldsymbol{\Theta})$ can be expressed as

$$\Sigma(\boldsymbol{K})\triangleq\mathrm{cov}(\mathrm{vec}(\boldsymbol{K}))=E\Big(\left[\mathrm{vec}(\boldsymbol{K})-\mathrm{vec}(\hat{\boldsymbol{K}})\right]\left[\mathrm{vec}(\boldsymbol{K})-\mathrm{vec}(\hat{\boldsymbol{K}})\right]^T\Big) \quad (45)$$

where E denotes the expectation operator and $\mathrm{vec}(\cdot)$ is an operator that stacks the columns of a matrix in a column vector. For example, if a matrix $\boldsymbol{A}=[\boldsymbol{a}_1\ \boldsymbol{a}_2]_{n\times2}$, then

$$\mathrm{vec}(\boldsymbol{A})=\begin{bmatrix}\boldsymbol{a}_1\\ \boldsymbol{a}_2\end{bmatrix} \quad (46)$$

Let $\Delta\boldsymbol{K}(\omega,\boldsymbol{\Theta})\triangleq\mathrm{vec}[\boldsymbol{K}\omega,\hat{\boldsymbol{\Theta}})-\boldsymbol{K}(\omega,\boldsymbol{\Theta})]_{4\times1}$. Then Eq. 42 can be written as

$$\Delta\boldsymbol{K}(\omega,\boldsymbol{\Theta})\approx\boldsymbol{K}_{\boldsymbol{\Theta}}(\hat{\boldsymbol{\Theta}}-\boldsymbol{\Theta}) \quad (47)$$

by a Taylor's series expansion. Therefore, the covariance matrix $\Sigma(\boldsymbol{K}(\boldsymbol{\omega},\boldsymbol{\Theta}))$ can be expressed by (Ljung, 1985)

$$\Sigma(\boldsymbol{K}(\omega,\boldsymbol{\Theta}))\approx\boldsymbol{K}_{\boldsymbol{\Theta}}\Sigma(\boldsymbol{\Theta})\boldsymbol{K}_{\boldsymbol{\Theta}}^T \quad (48)$$

Lemma 1 (Wahlberg and Ljung, 1992). Let $x\in R^n$, $\Sigma\in R^{n\times n}$ be positive definite, and assume that

$$x^T\Sigma^{-1}x\le1 \quad (49)$$

Consider $y=Ax$, where $y\in R^p$, $p\le n$, $A\in R^{p\times n}$ and A has full row rank. Then

$$y^T\left(A\Sigma A^T\right)^{-1}y\le1 \quad (50)$$

The $(1-\alpha)\times100\%$ confidence intervals can be derived from Lemma 1 with Eq. 41, 47 and 48 as

$$\Delta\boldsymbol{K}^T(\omega,\boldsymbol{\Theta})\left(\boldsymbol{K}_{\Theta}\Sigma(\boldsymbol{\Theta})\boldsymbol{K}_{\boldsymbol{\Theta}}^T\right)^{-1}\Delta\boldsymbol{K}(\omega,\boldsymbol{\Theta})\le\chi^2_{\alpha,q} \quad (51)$$

$$\Rightarrow\Delta\boldsymbol{K}^T(\omega,\boldsymbol{\Theta})\Sigma^{-1}(\boldsymbol{K}(\omega,\boldsymbol{\Theta}))\Delta\boldsymbol{K}(\omega,\boldsymbol{\Theta})\le\chi^2_{\alpha,q} \quad (52)$$

where $\boldsymbol{K}(\omega=0,\boldsymbol{\Theta})$ is the steady-state gain matrix. For simplicity of notation, assume $\omega=0$ and omit the $\boldsymbol{\Theta}$ dependence. Thus $\boldsymbol{K}$ and $\hat{\boldsymbol{K}}$ denote the steady-state gain matrix and its estimate.

Equation 23 for the steady-state RGA element λ_{ij} can be expressed as

$$\lambda_{ij}-\hat{\lambda}_{ij}\approx\boldsymbol{\lambda}_{ij,\boldsymbol{K}}\Delta\boldsymbol{K} \quad (53)$$

where $\boldsymbol{\lambda}_{ij,\boldsymbol{K}}=\left[\frac{\partial\lambda_{ij}}{\partial K_{11}}\ \frac{\partial\lambda_{ij}}{\partial K_{21}}\ \frac{\partial\lambda_{ij}}{\partial K_{12}}\ \frac{\partial\lambda_{ij}}{\partial K_{22}}\right]$. Therefore, the $(1-\alpha)\times100\%$ confidence intervals for λ_{ij} can be derived by using Lemma 1 with Eqs. 27 and 52

$$\left(\lambda_{ij}-\hat{\lambda}_{ij}\right)^T\left(\sigma^2_{\lambda_{ij}}\right)^{-1}\left(\lambda_{ij}-\hat{\lambda}_{ij}\right)\le\chi^2_{\alpha,q} \quad (54)$$

$$\Rightarrow\left(\lambda_{ij}-\hat{\lambda}_{ij}\right)^2\le\chi^2_{\alpha,q}\sigma^2_{\lambda_{ij}} \quad (55)$$

The extension of these confidence intervals to the dynamic RGA is straightforward because frequency-dependent $\boldsymbol{K}(\omega,\boldsymbol{\Theta})$ and $\Sigma[\boldsymbol{K}(\omega,\boldsymbol{\Theta})]$ are known.

A simulation example is considered to demonstrate how RGA confidence intervals can be established based on process identification.

Example 3. A simplified model for the Shell Heavy Oil Fractionator problem has been reported by Adusumilli et al. (1998)

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}\frac{4.05e^{-27s}}{50s+1} & \frac{1.77e^{-28s}}{60s+1}\\ \frac{5.39e^{-18s}}{50s+1} & \frac{5.72e^{-14s}}{60s+1}\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}+\begin{bmatrix}\frac{5.88e^{-27s}}{50s+1}\\ \frac{6.90e^{-15s}}{40s+1}\end{bmatrix}d \quad (56)$$

where y_1 and y_2 are the top and side end points, respectively; u_1 and u_2 are top and side draws, respectively.

The steady-state gain matrix is

$$\boldsymbol{K}=\begin{bmatrix}4.05 & 1.77\\ 5.39 & 5.72\end{bmatrix} \quad (57)$$

The corresponding λ_{11} value is $\lambda_{11}=1.70$ and the recommended control variable pairing is y_1-u_1/y_2-u_2.

A commercial software package ADAPTX (Larimore, 1996) was used for the process identification. It employs a subspace identification technique, Canonical Variate Analysis (CVA), to identify a linear, stochastic state space model and calculate the frequency domain response of the model. An uncertainty description in the form of 95% confidence limit is also calculated.

A MIMO PRBS input signal was designed as per Adusumilli et al. (1998) and used as the excitation signal for process identification. The sampling period was chosen as Δt

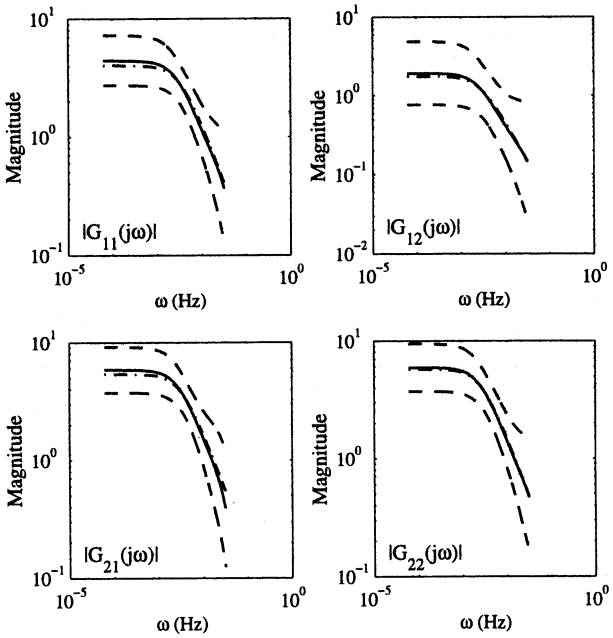

Figure 1. SNR = 3: frequency response with 95% confidence bands.

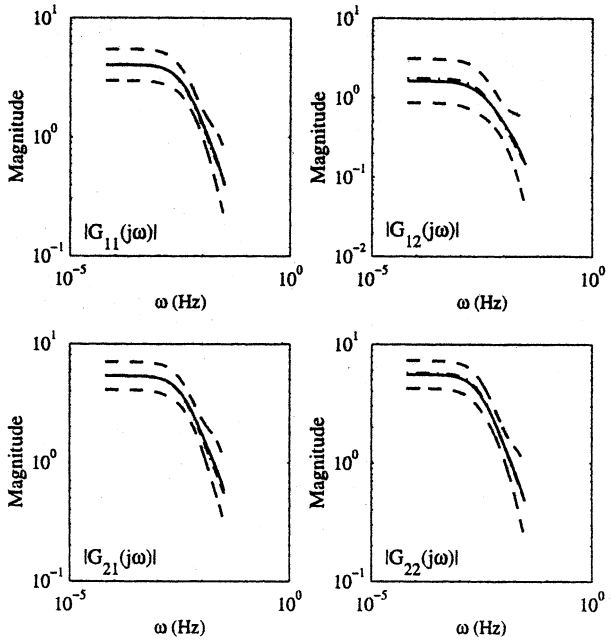

Figure 2. SNR = 10: frequency response with 95% confidence bands.

= 4 min. By using the ADAPTX software for CVA identification with a 2,500-point data set, a CVA based model was identified. Thus, the frequency-dependent gain matrix and its statistical uncertainty descriptions could be obtained. Then, the estimated RGA values and its uncertainty bounds were calculated.

Case 1. The signal-to-noise ratio for identification is $SNR = 3$.

Figure 1 shows the magnitude and 95% confidence intervals of the frequency response for the identified model. The solid line is the frequency response of the identified models and the dashed lines represent the 95% confidence bands. The dash-dot line is the frequency response of the true system. The estimated steady-state gains K_{ij} and the variances from the CVA identification are

$$\hat{\boldsymbol{K}} = \begin{bmatrix} 4.45 & 1.94 \\ 5.86 & 5.93 \end{bmatrix} \tag{58}$$

$$\sigma^2_{K_{11}} = 0.0933,\ \sigma^2_{K_{12}} = 0.105,\ \sigma^2_{K_{21}} = 0.129,\ \sigma^2_{K_{22}} = 0.145 \tag{59}$$

The nominal value of λ_{11} can be calculated using Eq. 8

$$\hat{\lambda}_{11} = 1.76 \tag{60}$$

The Cauchy-Schwartz inequality (Taylor, 1982)

$$|\mathrm{cov}(x,y)| \le \sigma_x \sigma_y \tag{61}$$

gives the upper bounds for the covariances cov (K_{ij}, K_{mn})

$$|\mathrm{cov}(K_{11},K_{12})| \le 0.099, \quad |cov(K_{11},K_{21})| \le 0.110,$$
$$|\mathrm{cov}(K_{11},K_{22})| \le 0.116 \tag{62}$$

$$|\mathrm{cov}(K_{12},K_{21})| \le 0.116, \quad |\mathrm{cov}(K_{12},K_{22})| \le 0.123,$$
$$|\mathrm{cov}(K_{21},K_{22})| \le 0.136 \tag{63}$$

Then, the upper bound for $\sigma^2_{\lambda_{11}}$ can be obtained from Eq. 27

$$\sigma^2_{\lambda_{11}} \le 0.0495. \tag{64}$$

For $\alpha = 0.05$, $\chi_{0.05,64} = 9.13$. Then using Eq. 55, the 95% confidence bands for λ_{11} are

$$-0.27 \le \lambda_{11} \le 3.79 \tag{65}$$

For this range of possible λ_{11} values, the recommended control variable pairing is ambiguous. The $-0.27 \le \lambda_{11} < 0.5$ range suggests the $y_1 - u_2/y_2 - u_1$ pairing, while the $y_1 - u_1/y_2 - u_2$ pairing is suggested for $0.5 < \lambda_{11} \le 3.79$. In view of the nominal value of $\hat{\lambda}_{11} = 1.76$, it seems more appropriate to select the $y_1 - u_1$ pairing. However, a more accurate identified model would be desirable in order to resolve this ambiguity.

Case 2. The signal-to-noise ratio is $SNR = 10$.

The identified frequency responses together with 95% confidence bands are shown in Figure 2. Also, the solid line is the frequency response of the identified model and the dashed lines are the 95% confidence bands. The dash-dot line is the frequency response of the true system.

The steady-state gain matrix and the variance for each element are

$$\hat{K} = \begin{bmatrix} 4.03 & 1.64 \\ 5.37 & 5.57 \end{bmatrix} \tag{66}$$

$$\sigma^2_{K_{11}} = 0.0253,\ \sigma^2_{K_{12}} = 0.0272,\ \sigma^2_{K_{21}} = 0.0348,\ \sigma^2_{K_{22}} = 0.0375 \tag{67}$$

Therefore, the nominal value of λ_{11} and its variance are

$$\hat{\lambda}_{11} = 1.64 \tag{68}$$

$$\sigma^2_{\lambda_{11}} \le 0.0121 \tag{69}$$

The corresponding 95% confidence interval for λ_{11} is

$$0.67 \le \lambda_{11} \le 2.62 \tag{70}$$

and the recommended RGA pairing is $y_1 - u_1/y_2 - u_2$ for the entire range.

Analysis for $n \times n$ Systems

The results obtained for 2×2 systems will now be extended to $n \times n$ systems. For a $n \times n$ system with the steady-state gain matrix

$$\boldsymbol{K} = \left[K_{ij}\right]_{n\times n} \tag{71}$$

the i,jth element of the RGA matrix $\boldsymbol{\Lambda}$ is (Grosdidier et al., 1985)

$$\lambda_{i,j} = (-1)^{i+j}\frac{K_{ij}\det(\boldsymbol{K}^{ij})}{\det(\boldsymbol{K})} \tag{72}$$

Here, $\boldsymbol{K}^{ij}$ denotes the submatrix that remains after the ith row and jth column of $\boldsymbol{K}$ are deleted. The partial derivative of λ_{ij} with respect to K_{kl} is given by

$$\frac{\partial\lambda_{ij}}{\partial K_{kl}} = \begin{cases} \dfrac{\lambda_{ij}(1-\lambda_{ij})}{K_{ij}} & i = k \text{ and } j = l \\ -\dfrac{\lambda_{ij}\lambda_{kl}}{K_{kl}} & i = k \text{ or } j = l \\ \dfrac{(-1)^{i+j+k+l}K_{ij}\det(\boldsymbol{K}^{ij,kl})}{\det(\boldsymbol{K})} - \dfrac{\lambda_{ij}\lambda_{kl}}{K_{kl}} & i \ne k \text{ and } j \ne l \end{cases} \tag{73}$$

Here, $\boldsymbol{K}^{ij,kl}$ denotes the submatrix of $\boldsymbol{K}$ with rows i and k and columns j and l removed. For the special case of $i = k$ and $j = l$, Eq. 73 is identical to Eq. 3 in Grosdidier et al. (1985).

Next, several different uncertainty descriptions for $n \times n$ problems are considered.

Worst-case bounds

Suppose that the uncertainty region for the steady state is given by $\hat{K}_{ij} \pm \Delta K_{ij}$. Then, the corresponding uncertainty bound for λ_{ij} is

$$|\hat{\lambda}_{ij} - \lambda_{ij}| \le \sum_{k=1}^{n}\sum_{l=1}^{n}\left|\left(\frac{\partial\lambda_{ij}}{\partial K_{kl}}\right)(\Delta K_{kl})\right| \tag{74}$$

Statisitical uncertainty description

Assume that $\hat{\boldsymbol{K}}$, the nominal steady-state gain matrix, and $\Sigma(\boldsymbol{K})$, the covariance matrix for each steady-state gain are known where

$$\Sigma(\boldsymbol{K}) \triangleq \text{cov}(\text{vec}(\boldsymbol{K})) = E\Big(\left[\text{vec}(\boldsymbol{K}) - \text{vec}(\hat{\boldsymbol{K}})\right]\left[\text{vec}(\boldsymbol{K}) - \text{vec}(\hat{\boldsymbol{K}})\right]^T\Big) \tag{75}$$

The nominal value of each λ_{ij} can be obtained from Eq. 72 with $\hat{\boldsymbol{K}}$ replacing $\boldsymbol{K}$. The variance is given by

$$\sigma^2_{\lambda_{ij}} = \left(\lambda_{ij,\boldsymbol{K}}\right)_{\hat{\boldsymbol{K}}}\Sigma(\boldsymbol{K})\left(\lambda_{ij,\boldsymbol{K}}\right)^T_{\hat{\boldsymbol{K}}} \tag{76}$$

where

$$\lambda_{ij,\boldsymbol{K}} = \left[\frac{\partial\lambda_{ij}}{\partial K_{11}}\ \frac{\partial\lambda_{ij}}{\partial K_{21}}\ \cdots\ \frac{\partial\lambda_{ij}}{\partial K_{nn}}\right] \tag{77}$$

denotes the row vector of the partial derivatives of λ_{ij} with respect to each element of the steady-state gain matrix $\boldsymbol{K}$. Equation 76 is the extension to $n \times n$ systems of the covariance formula Eq. 27 for 2×2 systems.

Example 4. Ogunnaike et al. (1983) considered a transfer function model for a pilot scale, binary distillation column used to separate ethanol and water

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \dfrac{0.66e^{-2.6s}}{6.7s+1} & \dfrac{-0.61e^{-3.5s}}{8.64s+1} & \dfrac{-0.0049e^{-s}}{9.06s+1} \\ \dfrac{1.11e^{-6.5s}}{3.25s+1} & \dfrac{-2.36e^{-3s}}{5s+1} & \dfrac{-0.012e^{-1.2s}}{7.09s+1} \\ \dfrac{-33.68e^{-9.2s}}{8.15s+1} & \dfrac{46.2e^{-9.4s}}{10.9s+1} & \dfrac{0.87(11.61s+1)e^{-s}}{(3.89s+1)(18.8s+1)} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{78}$$

where y_1 is overhead mol fraction of ethanol, y_2 is the mol fraction of ethanol in the side stream, and y_3 is temperature on Tray 19; u_1 is overhead reflux flow rate, u_2 is side stream draw-off rate, and u_3 is reboiler steam pressure.

The nominal steady-state gain matrix is given by

$$\hat{\boldsymbol{K}} = \begin{bmatrix} 0.66 & -0.61 & -0.0049 \\ 1.11 & -2.36 & -0.012 \\ -33.68 & 46.2 & 0.87 \end{bmatrix} \tag{79}$$

Therefore, the nominal RGA matrix is

$$\hat{\boldsymbol{\Lambda}} = \begin{bmatrix} 1.945 & -0.673 & -0.272 \\ -0.664 & 1.899 & -0.235 \\ -0.281 & -0.225 & 1.506 \end{bmatrix} \tag{80}$$

This RGA matrix suggests the diagonal controller pairing.

Assume that the uncertainty for each steady-state gain can be expressed as

$$|\Delta K_{ij}| \le \alpha |\hat{K}_{ij}| \tag{81}$$

Case 1: $\alpha = 0.01$. The uncertainty range for Λ is

$$\begin{bmatrix} \boxed{1.92 \le \lambda_{11} \le 1.97} & -0.698 \le \lambda_{12} \le -0.650 & -0.291 \le \lambda_{13} \le -0.253 \\ -0.689 \le \lambda_{21} \le -0.641 & \boxed{1.87 \le \lambda_{22} \le 1.93} & -0.246 \le \lambda_{23} \le -0.223 \\ -0.300 \le \lambda_{31} \le -0.262 & -0.237 \le \lambda_{32} \le -0.214 & \boxed{1.50 \le \lambda_{33} \le 1.52} \end{bmatrix} \tag{82}$$

The recommended RGA controller pairing is obvious and unambiguous: $y_1 - u_1/y_2 - u_2/y_3 - u_3$.

Case 2: $\alpha = 0.1$. The uncertainty range for Λ is

$$\begin{bmatrix} \boxed{1.65 \le \lambda_{11} \le 2.24} & -0.914 \le \lambda_{12} \le -0.434 & -0.459 \le \lambda_{13} \le -0.085 \\ -0.902 \le \lambda_{21} \le -0.641 & \boxed{1.57 \le \lambda_{22} \le 2.23} & -0.351 \le \lambda_{23} \le -0.118 \\ -0.470 \le \lambda_{31} \le -0.092 & -0.342 \le \lambda_{32} \le -0.109 & \boxed{1.39 \le \lambda_{33} \le 1.62} \end{bmatrix} \tag{83}$$

Again, the preferred controller pairing is: $y_1 - u_1/y_2 - u_2/y_3 - u_3$.

Case 3: $\alpha = 0.25$. Now the uncertainty range for Λ is

$$\begin{bmatrix} \boxed{1.21 \le \lambda_{11} \le 2.97} & -1.56 \le \lambda_{12} \le 0.167 & -0.739 \le \lambda_{13} \le 0.382 \\ -1.54 \le \lambda_{21} \le 0.168 & \boxed{1.07 \le \lambda_{22} \le 3.06} & -0.668 \le \lambda_{23} \le 0.173 \\ -0.754 \le \lambda_{31} \le 0.380 & -0.654 \le \lambda_{32} \le 0.182 & \boxed{1.22 \le \lambda_{33} \le 1.91} \end{bmatrix} \tag{84}$$

The same controller pairing is suggested for this case.

Case 4: $\alpha = 0.5$. The uncertainty range for Λ is

$$\begin{bmatrix} \boxed{0.477 \le \lambda_{11} \le 3.41} & -1.87 \le \lambda_{12} \le 0.527 & -1.21 \le \lambda_{13} \le 0.662 \\ -1.85 \le \lambda_{21} \le 0.524 & \boxed{0.244 \le \lambda_{22} \le 3.55} & -0.818 \le \lambda_{23} \le 0.348 \\ -1.23 \le \lambda_{31} \le 0.664 & -0.807 \le \lambda_{32} \le 0.356 & \boxed{0.931 \le \lambda_{33} \le 2.08} \end{bmatrix} \tag{85}$$

Based on the uncertainty ranges of λ_{ij}, the only appropriate pairing for the entire uncertainty ranges is: $y_1 - u_1/y_2 - u_2/y_3 - u_3$.

By calculating the uncertainty bounds of the RGA matrix for the entire uncertainty range of the plant model, we can check whether a single control structure selection is appropriate for all possible plants. Therefore, the robustness of the control structure is achieved.

Conclusions

Analytical expressions for RGA uncertainty bounds have been derived for 2×2 and $n \times n$ control problems. Two characterizations of process model uncertainty have been considered: worst case bounds and the types of statistical descriptions that are readily obtained during process identification. Several simulation examples illustrate the new results.

The new RGA uncertainty bounds provide useful information concerning the recommended control structure and the uncertainty associated with the recommendation. Furthermore, RGA uncertainty analysis can be used to estimate the maximum degree of uncertainty in the plant model that will not affect the recommended controller pairing. This useful

information provides insight concerning the required accuracy of the plant model and how much the operating condition can change before the controller pairing decision becomes ambiguous.

Although only the steady-state RGA has been considered in this article, all of the results can be directly extended to the frequency-dependent RGA matrix in order to gain information about dynamic behavior.

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Manuscript received May 4, 2001, and revision received Aug. 30, 2001.